On the max-plus algebra of non-negative exponent matrices

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An integer $n \times n$-matrix $A = (\alpha_{pq})$ is called exponent if all its diagonal entries are equal to zero and for all possible i, j and k the inequality $\alpha_{ij} + \alpha_{jk} \geqslant \alpha_{ik}$ holds. The study of exponent matrices is important because of their crucial role in the theory of tiled orders.

We show that the set $\mathcal{T}$ of minimal non-negative exponent $n \times n$-matrices can be described as follows. The matrix $T = (t_{ij}) \in \mathcal{E}_n$ belongs to $\mathcal{T}$ if and only if $t_{ij} \in \{0, 1\}$ for all i, j and there exists a proper subset $\mathcal{I}$ of $\{1, \ldots, n\}$ such that $t_{ij} = 1$ is equivalent to $i \in \mathcal{I}$ and $j \notin \mathcal{I}$.

Let $\oplus$ be the element-wise maximum of matrices and let $\otimes$ be a sum of matrices. Clearly, $A \otimes (B \oplus C) = (A \otimes B) \oplus (A \otimes C)$ for all A, B, $C \in \mathcal{E}_n$, whence $\mathcal{E}_n$ can be considered as an algebra $(\mathcal{E}_n, \oplus, \otimes)$, with respect to operations $\oplus$ and $\otimes$.

We prove the following result.

Theorem. *For any non-zero $A \in \mathcal{E}_n$ there exist a decomposition*

$$A = B_1 \otimes \ldots \otimes B_l \oplus \ldots \oplus C_1 \otimes \ldots \otimes C_m,$$

where all matrices $B_1, \ldots, C_m$ belong to $\mathcal{T}$ and as usual $\otimes$ performed prior to $\oplus$.

Thus, $\mathcal{T}$ can be considered as a basis of $(\mathcal{E}_n, \oplus, \otimes)$. This basis is unique. Nevertheless, there is no uniqueness of the decomposition of $A \in (\mathcal{E}_n, \oplus, \otimes)$ into the max-plus expression of matrices from $\mathcal{T}$.

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