Counting arithmetical structures of a graph and their sandpile groups.

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Given a graph $G = (V, E)$ and $\mathbf{d} \in \mathbb{N}$, the *Laplacian matrix* of the pair $(G, \mathbf{d})$ is the square matrix given by

$$L(G, \mathbf{d})_{u,v} = \begin{cases} \mathbf{d}_u & \text{if } u = v, \\ -m_{uv} & \text{if } u \neq v, \end{cases}$$

where m_{uv} is the number of edges between u and v. An *arithmetical structure* of G is a pair $(\mathbf{d}, \mathbf{r})$ such that $(\mathbf{d}, \mathbf{r}) \in \mathbb{N}_+^V \times \mathbb{N}_+^V$, $\gcd(\mathbf{r}_v \,|\, v \in V(G)) = 1$ and

$$L(G, \mathbf{d})\mathbf{r}^t = \mathbf{0}^t.$$

The concept of arithmetical graphs was introduced by Lorenzini as some intersection matrices that arise in the study of degenerating curves in algebraic geometry. If G is strongly connected, then

$$\mathcal{A}(G) = \{(\mathbf{d}, \mathbf{r}) \in \mathbb{N}_+^{V(G)} \times \mathbb{N}_+^{V(G)} \,|\, (\mathbf{d}, \mathbf{r}) \text{ is an arithmetical structure of } G\}.$$

is finite. Our goal is to describe and count the arithmetical structures and their associated sandpile groups of some simple graph, like the path, cycle, complete, etc. For instance we prove that the number of arithmetical structures of a path P_n with n vertices is equal to the Catalan number C_{n-1}.