Fuzzy Neighborhood Semantics

Ricardo Oscar Rodriguez
Departamento de Computación. FCEyN. Universidad de Buenos Aires, Argentina
ricardo@dc.uba.ar

Our starting point is that the framework of classical logic is not enough to reason with vague concepts or with modal notions such as belief, uncertainty, knowledge, obligations, time, etc. Many-valued logical systems under the umbrella of mathematical fuzzy logic (in the sense of Hájek [1]) appear as a suitable logical framework to formalize reasoning with vague or gradual predicates, while a variety of modal logics address the logical formalization to reason about different notions as the ones mentioned above. Therefore, if one is interested in a logical account of both vagueness and some sorts of modality, one is led to study systems of many-valued modal logic.

The basic idea of this presentation is to systematically introduce modal extensions of many-valued or fuzzy logics. These logics, under different forms and contexts, have appeared in the literature for different reasoning modeling purposes. For instance, in [2], Fitting introduces a modal logic on logics valued on finite Heyting algebras, and provides a satisfactory justification to study such modal systems to deal with opinions of experts with a dominance relationship among them. In [3] and [4], the authors have proposed to extend Gödel fuzzy logic with modal operators. They provide a systematic study of this Gödal modal logic, which has been completmented in [5]. In [6], a detailed description of many-valued modal logics (with a necessity operator) over finite residuated lattice is proposed. In [7], a modal extension of Lukasiewicz logic is developed following an algebraic approach. Finally, in [8], a general approach to modal expansions of t-norm based logics is also introduced with the help of rational constants and possibly infinitary inference rules.

In most of these mentioned papers, many-valued modal logics are endowed with a Kripke-style semantics, generalizing the classical one, where propositions at each possible world, and possibly accessibility relations between worlds as well, are valued in a residuated lattice. The natural next step in this line of research is to axiomatize such semantics. However, this has turned apparently to be a considerable overall challenge because it is difficult to transfer some usual techniques from Boolean algebras to residuated lattices. For instance, the K axiom $(\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi))$ plays a central role in the construction of the canonical models in order to prove completeness in the classical case. However, except for either Gödel modal logic or many-valued modal logics defined from Kripke frames with crisp accessibility relations, the K axiom is not sound.

In order to overcome this difficulty, we propose to study an alternative semantics which is a generalization of the classical *neighborhood semantics*. This will be elaborated based on two preliminary workshop papers by the same authors ([9] and [10]). At this moment, it is worth mentioning some works from other authors which consider a generalization of neighborhood semantics in the same way we have done it. Namely, Kroupa and Teheux consider in [11] a neighborhood semantics for playable L_n-valued effectivity function. They want to characterize the notion of coalitional effectivity within game form models. Also we must mention a very recent paper by Cintula et al. ([12]) where the authors explore a fuzzified version of the classical neighborhood semantics and prove a relationship between fuzzy Kripke and neighborhood semantics in a very precise way (much better than the one proposed in our previous work). In fact, the authors of this paper propose to attack the problem of characterizing the modal extensions of MTL logics under a neighborhood semantics with algebraic tools. According to their algebraic approach, they characterize a global MTL modal logic, leaving open the case of characterizing the local consequence relation.

In summary, in this presentation, we will mainly focus on the development of a theoretical and general framework. Considering our motivation, our main goal, at large, is a systematic presentation of the minimum many-valued modal logics and their extensions. In this sense, we will firstly present minimum

many-valued modal logics with necessity and possibility operators, $\Box, \Diamond$, defined on top of logics of residuated lattices under a neighborhood semantics.

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