Monadic Gödel algebras are functional

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Monadic Heyting algebras where introduced by Monteiro and Varsavsky (1957) as an algebraic counterpart of the one variable fragment of quantified monadic intuitionistic logic, generalizing monadic Boolean algebras introduced by Halmos (1955) with a similar purpose for classical logic. They were found to interpret also the intuitionistic analogue of the modal system $S5$ and have been extensively studied since. Answering a question put by Monteiro, Bezhanishvili and Harding (2002) proved that any monadic Heyting algebra is functional; that is, it may be embedded in an algebra of functions $(H^X, \triangle, \nabla)$, where H is a complete Boolean algebra, the Heyting operations are defined pointwise, and the monadic operators $\triangle$ and ∇ are interpreted as $\triangle f = \inf_{x\in X} f(x)$, $\nabla f = \sup_{x\in X} f(x)$, respectively. Apart from the variety of boolean algebras for which Halmos proved a similar result, the situation is unknown for other familiar varieties of Heyting algebras. We solve this problem for monadic Gödel algebras, which interpret the one variable fragment of monadic predicate logic with values in the standard Gödel chain [0,1] (or in all linear Heyting algebras, Horn, 1969; Baaz et al, 2007). These are the monadic Heyting algebras satisfying the prelinearity axiom and the identity $\triangle(\triangle a \vee b) = \triangle a \vee \triangle b$ corresponding to the quantifier shift law $\forall x(\forall x\varphi \vee \psi) \leftrightarrow \forall x\varphi \vee \forall x\psi$, and constitute also the algebraic semantics of the Gödel analogue of S5 (Hájek, 2010, C. & Rodríguez 2012). Any monadic Gödel algebra may be embedded in an algebra of functions H^X where H is a complete Gödel algebra. For countable algebras, H may be chosen to be $[0, 1]$.