Hochschild homology and cohomology of down-up algebras

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We have computed Hochschild homology and cohomology of homogeneous down-up algebras in the generic case and in the Calabi-Yau case.

Let K be a fixed field. Given parameters $(\alpha, \beta, \gamma) \in K^3$, the associated down-up algebra $A(\alpha, \beta, \gamma)$ is defined as the quotient of the free associative algebra $K\langle u, d\rangle$ by the ideal generated by the relations

$$\begin{aligned} &d^2u - (\alpha dud + \beta ud^2 + \gamma d), \\ &du^2 - (\alpha udu + \beta u^2 d + \gamma u). \end{aligned} \tag{1}$$

This family of algebras was introduced by G. Benkart and T. Roby. As typical examples we have that $A(2, -1, 0)$ is isomorphic to the enveloping algebra of the Heisenberg-Lie algebra of dimension 3, and, for $\gamma \neq 0$, $A(2, -1, \gamma)$ is isomorphic to the enveloping algebra of $\mathfrak{sl}(2, K)$. Moreover, Benkart proved that any down-up algebra such that $(\alpha, \beta) \neq (0, 0)$ is isomorphic to one of Witten's 7-parameter deformations of $U(sl(2, K))$.

E. Kirkman, I. Musson and D. Passman proved that $A(\alpha, \beta, \gamma)$ is noetherian if and only it is a domain, which is tantamount to the fact that the subalgebra of $A(\alpha, \beta, \gamma)$ generated by ud and du is a polynomial algebra in two indeterminates, that in turn is equivalent to $\beta \neq 0$. Under either of the previous situations, $A(\alpha, \beta, \gamma)$ is Auslander regular and its global dimension is 3. On the other hand, it was proved by Cassidy and Shelton that, if K is algebraically closed, then the global dimension of $A(\alpha, \beta, \gamma)$ is always 3. Moreover, Benkart and Roby also proved that the Gelfand-Kirillov dimension of a down-up algebra is 3, independently of the parameters.

If $\gamma = 0$, the down-up algebra can be regarded as nonnegatively graded, where the degree of u and d is 1. In this case, the algebra is 3-Koszul and Artin-Schelter regular.

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